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THE PROBLEM OF RELATIVE MAXIMA OR MINIMA UNDER A NEW POINT OF VIEW.*

By MR. CHAS. H. KUMMELL, Washington, D. C.

Price, in his Infinitesimal Calculus, Vol. I, pp. 235 to 278, gives a treatment of this problem, complete in every respect save one; for on page 274 he dismisses the problem of discriminating between a maximum or minimum, in case of a relative maximum or minimum, by saying: "The algebraical criterion for discriminating between a maximum and a minimum is too complicated to be of any service even in this particular case."† While admitting this difficulty, the reader should however, not be left in any doubt as to the method to be followed, if, in spite of this difficulty, he wishes to make the discrimination. Since Price gives a very elaborate treatment of this discrimination problem for a function of n variables we may of course, by eliminating m variables by m conditions, and thus forming a function of $n - m$ variables, apply his methods to this latter. This evasion of the method of relative maxima or minima is usually highly inelegant, and there should be a discriminating method immediately applicable.

Suppose we require the conditions for maximum or minimum of the function

$$F(x_1, x_2, \ldots x_n), \tag{1}$$

subject to the m conditions

$$\begin{aligned} f_1\ (x_1, x_2, \ldots x_n) &= 0, \\ f_2\ (x_1, x_2, \ldots x_n) &= 0, \\ \cdots\cdots & \\ f_m (x_1, x_2, \ldots x_n) &= 0. \end{aligned} \tag{2}$$

If we form the new function

$$u = F + c_1 f_1 + c_2 f_2 + \ldots + c_m f_m, \tag{3}$$

it is at once apparent that, by virtue of (2), u is identical with F, although it has

*Read before the Mathematical Section of the Philosophical Society of Washington.
†It is the case of but one condition between the variables.

$n + m$ variables. If we subject this new function u to the conditions of a maximum or minimum, we have

$$0 = \frac{du}{dx_1} = \frac{du}{dx_2} = \ldots = \frac{du}{dx_n}, \tag{4}$$

$$\left.\begin{aligned} 0 &= \frac{du}{dc_1} = f_1, \\ 0 &= \frac{du}{dc_2} = f_2, \\ &\cdots\cdots \\ 0 &= \frac{du}{dc_m} = f_m; \end{aligned}\right. \tag{2'}$$

and since the latter m conditions are identical with the given conditions (2), it follows that the function u *need not be considered subject to any conditions.* In using the function u in place of F we have therefore the problem to find the conditions of maximum or minimum of a function of $n + m$ variables, and since Price has given, pp. 252–265, a complete treatment of the problem of discriminating between a maximum or a minimum of a function of any number of variables which are not subject to any condition, we have thus at the same time the required directly applicable method for relative maxima and minima.

To apply these principles to an easy problem, let it be required to divide a given number a into two parts x and y so that $x^m y^n$ shall be a maximum (or minimum?). We may take for F in this case the logarithm of $x^m y^n$; and since

$$x + y - a = 0,$$

we have
$$u = mlx + nly + c(a - x - y);$$

$$0 = \frac{du}{dx} = \frac{m}{x} - c, \qquad \frac{d^2u}{dx^2} = -\frac{m}{x^2}, \qquad \frac{d^2u}{dydc} = -1;$$

$$0 = \frac{du}{dy} = \frac{n}{y} - c, \qquad \frac{d^2u}{dy^2} = -\frac{n}{y^2}, \qquad \frac{d^2u}{dcdx} = -1;$$

$$0 = \frac{du}{dc} = a - x - y, \qquad \frac{d^2u}{dc^2} = 0, \qquad \frac{d^2u}{dxdy} = 0.$$

We obtain
$$x = \frac{ma}{m+n}, \qquad y = \frac{na}{m+n};$$

and we have to decide whether $\left(\frac{ma}{m+n}\right)^m \left(\frac{na}{m+n}\right)^n$, or else $m^m n^n \left(\frac{a}{m+n}\right)^{m+n}$,

is a maximum or a minimum. To form the discriminating cubic we have

$$\begin{aligned}
\left(\frac{m}{x^2}+\theta\right) dx &+ \quad 0 \,.\, dy &+ \quad dc &= 0,\\
0 \,.\, dx &+ \left(\frac{n}{y^2}+\theta\right) dy &+ \quad dc &= 0,\\
dx &+ \quad dy &+ (0+\theta)\, dc &= 0;
\end{aligned}$$

whence
$$0 = \begin{vmatrix} \dfrac{m}{x^2}+\theta, & 0, & 1 \\ 0, & \dfrac{n}{y^2}+\theta, & 1 \\ 1, & 1, & \theta \end{vmatrix}$$

$$= \left(\frac{m}{x^2}+\theta\right)\left(\frac{n}{y^2}+\theta\right)\theta - \left(\frac{m}{x^2}+\theta\right) - \left(\frac{n}{y^2}+\theta\right)$$

$$= \theta^3 + \left(\frac{m}{x^2}+\frac{n}{y^2}\right)\theta^2 + \left(\frac{mn}{x^2y^2}-2\right)\theta - \frac{m}{x^2} - \frac{n}{y^2}.$$

Because the 2d and 4th terms are always of opposite sign, it will be impossible that the three roots of this discriminating cubic can be of the same sign; consequently there can be no total maximum with respect to the three variables x, y, c. Leaving the third variable c out of consideration, we have for a partial maximum or minimum the discriminating quadratic

$$\left(\frac{m}{x^2}+\theta\right)\left(\frac{n}{y^2}+\theta\right) = 0 = \theta^2 + \left(\frac{m}{x^2}+\frac{n}{y^2}\right)\theta + \frac{mn}{x^2y^2}.$$

The roots of this equation are always negative if m and n are positive; hence the condition for a partial maximum with respect to x and y are fulfilled, since then likewise d^2u/dx^2 and d^2u/dy^2 are negative.

As in this case, so in others, it will be unnecessary to consider the total maximum or minimum, but only the partial maximum or minimum for the given variables.

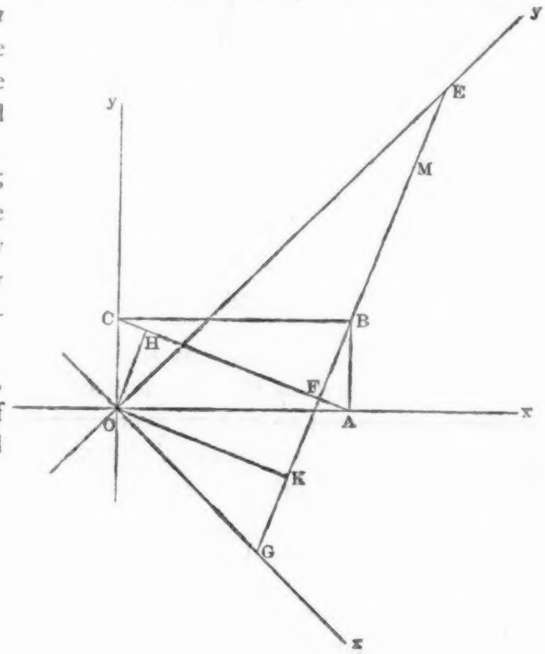
A METHOD OF FINDING THE EVOLUTE OF THE FOUR-CUSPED HYPOCYCLOID.

By Prof. R. H. Graves, Chapel Hill, N. C.

The following is a method, based on the Theory of Roulettes, of proving the well-known result, that the evolute of a four-cusped hypocycloid is also a four-cusped hypocycloid. Let $AC = a$ move with its extremities on the rectangular axes Ox and Oy. The envelope of AC is the four-cusped hypocycloid.

Complete the rectangle $OABC$; draw Ox' and Oy' bisecting the angles between Ox and Oy; draw $EBFG$ perpendicular to AC; draw OK and OH parallel and perpendicular to AC; let $CAO = \theta$.

B is the instantaneous centre, BF is the normal to the envelope of AC, and its envelope is the required evolute.



$$ABF = \theta,$$
$$FBC = 90^\circ - \theta,$$
$$BEO = 45^\circ - \theta,$$
$$OGK = 45^\circ + \theta.$$

$$OK = HF = a - 2a \sin^2 \theta = a \cos 2\theta.$$

$$EG = OK \operatorname{cosec} OGK \operatorname{cosec} BEO = a \cos 2\theta \operatorname{cosec} (45^\circ + \theta) \operatorname{cosec} (45^\circ - \theta).$$

Hence, by reduction, $$EG = 2a.$$

Since EG is of constant length, and its extremities move on two rectangular axes, its envelope must be a four-cusped hypocycloid, which is the required evolute.

Remark.—If M is the point where EG touches its envelope, $BM = BK$.

For, at the point (x, y) of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, the radius of curvature is $3\,(axy)^{\frac{1}{3}}$; and the perpendicular from the origin on the tangent is $(axy)^{\frac{1}{3}}$.

Or, it follows from the formula connecting the segments into which the radius of curvature of the hypocycloid is divided by the instantaneous centre. (See Williamson's Differential Calculus, Art. 281.)

ON CERTAIN SINGULARITIES OF THE HESSIANS OF THE CUBIC AND THE QUARTIC.

By Mr. William E. Heal, Marion, Ind.

The Hessian of a curve of the nth degree,

$$u = \varphi(x, y, z) = 0,$$

is a curve of the $3(n-2)$th degree,

$$\begin{vmatrix} \dfrac{d^2u}{dx^2}, & \dfrac{d^2u}{dxdy}, & \dfrac{d^2u}{dxdz} \\ \dfrac{d^2u}{dydx}, & \dfrac{d^2u}{dy^2}, & \dfrac{d^2u}{dydz} \\ \dfrac{d^2u}{dzdx}, & \dfrac{d^2u}{dzdy}, & \dfrac{d^2u}{dz^2} \end{vmatrix} = 0.$$

It is easy to prove* that the Hessian passes through the points of inflexion of u, and also† that if a point be a multiple point of order k on u, it is a multiple point of order $3k-4$ on the Hessian. The question may then be proposed: Given the proposition that a point is a singular point on u, what is the nature of its singularity on the Hessian? I shall attempt to answer this question for the cases of cubics and quartics. By placing $z = 1$ and taking the singular point as the origin, the discussion will be greatly simplified, as the equation will then lose one or more dimensions.

THE CUBIC.

The general equation of the cubic may be written

$$ax^3 + by^3 + cz^3 + 3a'x^2y + 3a''x^2z + 3b'y^2x + 3b''y^2z + 3c'z^2x \\ + 3c''z^2y + 6mxyz = 0.$$

The equation of its Hessian is‡

$$\begin{aligned} &(ab'c' - am^2 + 2a'a''m - a''^2b' - a'^2c')\,x^3 \\ &+ (a'bc'' - bm^2 + 2b'b''m - a'b''^2 - b'^2c'')\,y^3 \\ &+ (a''b''c - cm^2 + 2c'c''m - a''c''^2 - b''c'^2)\,z^3 \\ &+ (abc' - 2ab''m + ab'c'' - a''^2b + a'm^2 - a'b'c' + 2a'a''b'' - a'^2c'')x^2y \\ &+ \begin{pmatrix} ab'c - 2ac''m + ab''c' - a'^2c + a''m^2 \\ - a''b'c' + 2a'a''c'' - a''^2b'' \end{pmatrix} x^2z \end{aligned}$$

*Salmon's Higher Plane Curves, 3d Edition, Art. 74.
†Same, Art. 75.
‡Same, Art. 218.

$$+ (abc'' - 2a''bm + a'bc' - ab'^{\prime 2} + b'm^2 - a'b'c'' + 2a''b'b'' - b'^2c')\,y^2x$$
$$+ (a'bc - 2bc'm + a''bc'' - b'^2c + b''m^2 - a'b''c'' + 2b'b''c' - a''b''^2)\,y^2z$$
$$+ (ab''c - 2a'cm + a''b'c - ac''^2 + c'm^2 - a''b''c' + 2a'c'c'' - b'c'^2)\,z^2x$$
$$+ (a''bc - 2b'cm + a'b''c - bc'^2 + c''m^2 - a''b''c'' + 2b'c'c'' - a'c''^2)\,z^2y$$
$$+ \left\{ \begin{matrix} abc - ab''c'' - a''bc' - a'b'c + 2m^3 - 2b'c'm \\ - 2a'c''m - 2a''b''m + 3a'b''c' + 3a''bc'' \end{matrix} \right\} xyz = 0.$$

If the origin is a double point on the cubic, all terms of a degree below the second vanish; that is

$$c = c' = c'' = 0.$$

The cubic is then

$$ax^3 + 3a'x^2y + 3b'xy^2 + by^3 + 3a''x^2 + 6mxy + 3b''y^2 = 0.$$

The equation to the tangents at the double point found by equating to zero the terms of the second degree is

$$a''x^2 + 2mxy + b''y^2 = 0.$$

The equation of the Hessian becomes

$$\begin{aligned} &(2a'a''m - am^2 - a''^2b')\,x^3 \\ &+ (a'm^2 - a''^2b - 2ab''m + 2a'a''b'')\,x^2y \\ &+ (b'm^2 - ab''^2 - 2a''bm + 2a''b'b'')\,xy^2 \\ &+ (2b'b''m - bm^2 - a'b''^2)\,y^3 \\ &+ (m^2 - a''b'')\,(a''x^2 + 2mxy + b''y^2) = 0. \end{aligned}$$

The origin is a double point on the Hessian, as has been already stated; but also the tangents at the double point on the Hessian coincide with the tangents at the double point on the cubic, for their equation is

$$a''x^2 + 2mxy + b''y^2 = 0.$$

The proof of this last proposition given by Salmon (Art. 75) seems to apply only to the case of real tangents.

The Hessian is, then, crunodal or acnodal according as the tangents at the double point on u are real or imaginary. We might infer that the Hessian of a cuspidal cubic is a cuspidal cubic, but such is not the case. In fact, it may be proved generally (Salmon, Art. 77) that a cusp on u is a triple point on the Hessian. Following our line of proof, the condition that the tangents at the double point on u may coincide is

$$m^2 - a''b'' = 0;$$

but reverting to the equation of the Hessian, we see that this quantity is a factor of all terms of the second degree, and therefore the equation contains no terms of lower degree than the third. The origin is then a triple point. But a cubic cannot have a triple point unless it break up into three right lines. Writing the equation of the Hessian in the form

$$(a''x^2 + 2mxy + b''y^2)\left[(2a'm - ab'' - a''b')\,x + (2b'm - a''b - a'b'')\,y\right] = 0,$$

we infer that the Hessian of a cuspidal cubic is three right lines, two of which coincide with the tangent at the cusp on u. For example, the Hessian of the cissoid

$$x\,(x^2 + y^2) = 2ry^2,$$

and also the semi-cubical parabola

$$ry^2 = x^3,$$

is

$$xy^2 = 0,$$

which represents three right lines, two coinciding with the axis of x and the other with the axis of y.

Suppose that

$$a'' = c = c' = 0.$$

The origin is a point of inflexion on u, and $y = 0$, the tangent at it. The Hessian is

$$\begin{aligned}
&- am^2x^3 \\
&+ (ab'c'' - 2ab''m + a'm^2 - a'^2c'')\,x^2y \\
&+ (abc'' - ab''^2 + b'm^2 - a'b'c'')\,xy^2 \\
&+ (a'bc'' - bm^2 + 2b'b''m - a'b''^2 - b'^2c'')\,y^3 \\
&- 2ac''mx^2 + (2m^3 - ab''c'' - 2a'c''m)\,xy \\
&+ b''\,(m^2 - a'c'')\,y^2 \\
&- ac''^2x + c''\,(m^2 - a'c'')\,y = 0.
\end{aligned}$$

The origin is evidently a point on the Hessian, but we will further show that it is a point of inflexion.

Turn the axes through an angle θ such that

$$\tan\theta = \frac{ac''}{m^2 - a'c''}.$$

We must have

$$X = c''\,(m^2 - a'c'')\,x - ac''^2y,$$
$$Y = ac''^2x + c''\,(m^2 - a'c'')\,y.$$

It will then be found that the coefficients of x and of x^2 vanish; i. e. the origin is a point of inflexion, and the new axis of x the tangent at it.

THE QUARTIC.

We write the general equation of the quartic in the form

$$
\begin{aligned}
&ax^4 + by^4 + cz^4 \\
&+ 6fy^2z^2 + 6gz^2x^2 + 6hx^2y^2 \\
&+ 12lx^2yz + 12my^2zx + 12nz^2xy \\
&+ 4a'x^3y + 4a''x^3z + 4b'y^3x + 4b''y^3z + 4c'z^3x + 4c''z^3y = 0.
\end{aligned}
$$

The equation of the Hessian is not given in any mathematical work in my possession. According to my calculation it is

$$
\begin{aligned}
&(agh + 2a'a''l - al^2 - a''^2h - a'^2g)\,x^6 \\
&+ (bfh + 2b'b''m - bm^2 - b''^2h - b'^2f)\,y^6 \\
&+ (cfg + 2c'c''n - cn^2 - c''^2g - c'^2f)\,z^6 \\
&+ 2\,(ab'g + ahn + a'l^2 + 2a'a''m - 2alm - a'gh - a''^2b' - a'^2n)\,x^5y \\
&+ 2\,(a'bf + bhn + b'm^2 + 2b'b''l - 2blm - b'fh - a'b''^2 - b'^2n)\,xy^5 \\
&+ 2\,(bc''h + bfl + b''m^2 + 2b'b''n - 2bmn - b''fh - b'^2c'' - b''^2l)\,y^5z \\
&+ 2\,(b''cg + cfl + c''n^2 + 2c'c''m - 2cmn - c''fg - b''c'^2 - c''^2l)\,z^5y \\
&+ 2\,(a''cf + cgm + c'n^2 + 2c'c''l - 2cln - c'fg - a''c''^2 - c'^2m)\,z^5x \\
&+ 2\,(ac'h + agm + a''l^2 + 2a'a''n - 2aln - a''gh - a'^2c' - a''^2m)\,x^5z \\
&+ \left(\begin{matrix} abg + afh - a''^2b - a'^2f + 2a'b'g + 2a'a''b'' + 2a'lm - 2ab''l \\ + 3hl^2 - 3gh^2 + 4ab'n - 4a'hn - 4am^2 + 6a''hm - 6a''b'l \end{matrix}\right) x^4y^2 \\
&+ \left(\begin{matrix} abf + bgh - ab''^2 - b'^2g + 2a'b'f + 2a''b'b'' + 2b'lm - 2a''bm \\ + 3hm^2 - 3fh^2 + 4a'bn - 4b'hn - 4bl^2 + 6b''hl - 6a'b''m \end{matrix}\right) x^2y^4 \\
&+ \left(\begin{matrix} bfg + bch - b'^2c - b''^2g + 2b''c''h + 2b'b''c' + 2b''mn - 2bc'm \\ + 3fm^2 - 3f^2h + 4bc''l - 4b''fl - 4bn^2 + 6b'fn - 6b'c''m \end{matrix}\right) y^4z^2 \\
&+ \left(\begin{matrix} cfh + bcg - bc'^2 - c''^2h + 2b''c''g + 2b'c'c'' + 2c''mn - 2b'cn \\ + 3fn^2 - 3f^2g + 4b''cl - 4c''fl - 4cm^2 + 6c'fm - 6b''c'n \end{matrix}\right) y^2z^4 \\
&+ \left(\begin{matrix} acf + cgh - ac''^2 - c'^2h + 2a''c'f + 2a'c'c'' + 2c'ln - 2a'cn \\ + 3gn^2 - 3fg^2 + 4a''cm - 4c'gm - 4cl^2 + 6c''gl - 6a''c''n \end{matrix}\right) z^4x^2 \\
&+ \left(\begin{matrix} ach + afg - a'^2c - a''^2f + 2a''c'h + 2a'a''c'' + 2a''ln - 2ac''l \\ + 3gl^2 - 3g^2h + 4ac'm - 4a''gm - 4an^2 + 6a'gn - 6a'c'l \end{matrix}\right) x^4z^2 \\
&+ 2\left(\begin{matrix} ab''g + ac''h - a''^2b'' - a'^2c'' + 2ab'c' + 2a'a''f \\ - 2a'c'h - 2amn - 2a''b'g - 2a'ln - 2a''lm \\ - 2afl + 3l^3 - 3ghl + 4a'gm + 4a''hn \end{matrix}\right) x^4yz
\end{aligned}
$$

$$+ 2 \left(\begin{array}{c} a''bf + bc'h - a''b''^2 - b'^2c' + 2a'bc'' + 2b'b''g \\ - 2b'c''h - 2bln - 2a'b''f - 2b'mn - 2b''lm \\ - 2bgm + 3m^3 - 3fhm + 4b'fl + 4b''hn \end{array}\right) y^4zx$$

$$+ 2 \left(\begin{array}{c} a'cf + b'cg - a'c''^2 - b'c'^2 + 2c'c''h + 2a''b''c \\ - 2b''c'g - 2clm - 2a''c''f - 2c'mn - 2c''ln \\ - 2chn + 3n^3 - 3fgn + 4c'fl + 4c''gm \end{array}\right) z^4xy$$

$$+ 2 \left(\begin{array}{c} abn + ab'f + a'bg - a'fh - b'gh + 2a'b'n \\ + 2a''b''h - 2a'm^2 - 2b'l^2 - 2ab''m \\ - 2a''bl - 3l^2n + 6hlm \end{array}\right) x^3y^3$$

$$+ 2 \left(\begin{array}{c} bcl + b''ch + bc''g - b''fg - c''fh + 2b'c'f + 2b''c''l \\ - 2b''n^2 - 2c''m^2 - 2bc'n - 2b'cm - 3f^2l + 6fmn \end{array}\right) y^3z^3$$

$$+ 2 \left(\begin{array}{c} acm + ac'f + a''ch - c'gh - a''fg + 2a''c'm + 2a'c''g \\ - 2c'l^2 - 2a''n^2 - 2ac''n - 2a'cl - 3g^2m + 6gln \end{array}\right) z^3x^3$$

$$+ 2 \left(\begin{array}{c} abc' - a''bg + 2ab'c'' + 2a''b'n + 2a'b'c' - 2a''m^2 \\ - 2a'c''h - 2a'mn - 2a'fl + 3ghm - 3afm \\ - 3c'l^2 + 4a'b''g - 4a''b''l + 5a''fh + 6l^2m - 6b'gl \end{array}\right) x^3y^2z$$

$$+ 2 \left(\begin{array}{c} ab'c - a'ch + 2ab''c' + 2a'c'm + 2a''b'c' - 2a'n^2 \\ - 2a''b''g - 2a''mn - 2a''fl + 3ghn - 3afn \\ - 3b'g^2 + 4a''c''h - 4a'c''l + 5a'fg + 6l^2n - 6c'hl \end{array}\right) x^3yz^2$$

$$+ 2 \left(\begin{array}{c} abc'' - ab''f + 2a'bc' + 2a'b''n + 2a'b'c'' - 2b''l^2 \\ - 2b'c'h - 2b'ln - 2b'gm + 3fhl - 3bgl - 3c''h^2 \\ + 4a''b'f - 4a''b''m + 5b''gh + 6lm^2 - 6a'fm \end{array}\right) y^3x^2z$$

$$+ 2 \left(\begin{array}{c} a'bc - b'ch + 2a''bc'' + 2b'c''l + 2a'b''c'' - 2b'n^2 \\ - 2b''ln - 2a''b''f - 2b''gm + 3fhn - 3bgn - 3a'f^2 \\ + 4b''c'h - 4b'c'm + 5b'fg + 6m^2n - 6c''hm \end{array}\right) y^3xz^2$$

$$+ 2 \left(\begin{array}{c} ab''c - ac''f + 2a''b'c + 2a''c''m + 2a''b''c' - 2c''l^2 \\ - 2c'hn - 2b'c'g - 2c'lm + 3fgl - 3chl - 3b''g^2 \\ + 4a'c'f - 4a'c''n + 5c''gh + 6ln^2 - 6a''fn \end{array}\right) z^3x^2y$$

$$+ 2 \left(\begin{array}{c} a''bc - bc'g + 2a''b''c'' + 2b''c'l + 2a'b''c - 2c'm^2 \\ - 2c''hm - 2a'c''f - 2c''lm + 3fgm - 3chm - 3a''f^2 \\ + 4b'c''g - 4b'c'n + 5c'fh + 6mn^2 - 6b''gn \end{array}\right) z^3y^2x$$

$$+\left\{\begin{array}{c} abc + 2ab''c'' + 2a''bc' + 2a'b'c - 3af^2 - 3ch^2 \\ - 3bg^2 - 6a''fm - 6b''gl - 6c''hl - 6a'fn \\ - 6b'gn - 6c'hm - 6a'c''m - 6b'c'l - 6a''b''n \\ + 10a'b''c' + 10a''b'c'' + 18fgh + 18lmn \end{array}\right\} x^2y^2z^2 = 0.$$

Let $$c = c' = c'' = 0.$$

The origin is a double point on the quartic, and the equation of the tangents at it is

$$gx^2 + 2nxy + fy^2 = 0.$$

The Hessian becomes

$$u_6 + u_5 + u_4 + u_3 + 3(n^2 - fg)(gx^2 + 2nxy + fy^2) = 0,$$

where u_n denotes the sum of the terms of the nth degree. The origin is also a double point on the Hessian unless $n^2 - fg = 0$.

The equation of the tangents to the Hessian at the double point is

$$gx^2 + 2nxy + fy^2 = 0,$$

and therefore the Hessian is acnodal or crunodal if the quartic is so. If $n^2 - fg = 0$, all terms below the third degree vanish, and the origin is a triple point on the Hessian. But this is the condition that the tangents at the double point on the quartic should be coincident; i. e. that the origin should be a cusp. The equation of the tangents at the triple point is

$$u_3 = (2gln - g^2m - a''fg)\,x^3 + (3fgl - b''g^2 - 2a''fn)\,x^2y$$
$$+ (3fgm - a''f^2 - 2b''gn)\,xy^2 + (2fmn - f^2l - b''fg)\,y^2 = 0.$$

If we write this equation in the form

$$(gx^2 + 2nxy + fy^2)\,[(2ln - gm - a''f)\,x + (2mn - fl - b''g)\,y] = 0,$$

we see that two tangents at the triple point coincide with the tangent at the cusp on the quartic.

Let $$c = c' = g = 0.$$

The origin is a point of inflexion on the quartic, and the case of the cubic suggests that it is also a point of inflexion on the Hessian. The Hessian is

$$\begin{aligned} &u_6 + u_5 + u_4 + u_3 \\ &- c''(ac'' + 6a''n)\,x^2 \\ &+ 2(3n^3 - a'c''^2 - 2a''c''f - 2c''ln)\,xy \\ &+ (2c''mn + 3fn^2 - c''^2h - 4c''fl)\,y^2 \\ &- a''c''^2x + c''(n^2 - c''l)\,y = 0. \end{aligned}$$

Turn the axes through an angle θ such that

$$\tan\theta = \frac{a''c''}{n^2 - c''l}.$$

We must have

$$X = c''(n^2 - c''l)\,x - a''c''^2 y,$$
$$Y = a''c''^2 + c''(n^2 - c''l).$$

It will then be found that the coefficient of x vanishes, but that of x^2 does not. The Hessian passes through the point of inflexion, but does not seem to present any singularity at that point.

Let

$$a'' = c = c' = g = 0.$$

The origin is now a point of undulation on the quartic, the tangent $y = 0$ meeting the curve in four consecutive points. The Hessian for this case is

$$\begin{aligned} &u_6 + u_5 + u_4 + u_3 \\ &- ac''^2x^2 + 2(3n^3 - a'c''^2 - 2c''ln)\,xy \\ &+ (2c''mn + 3fn^2 - c''^2h - 4c''fl)\,y^2 + c''(n^2 - c''l)\,y = 0. \end{aligned}$$

The tangent $y = 0$, at the point of undulation is also a tangent to the Hessian, but like the preceding case there is no singularity.

We will now consider singularities arising from the union of two or more simple singularities.

Let

$$a'' = c = c' = c'' = g = n = 0.$$

The origin is, then, a tacnode on the quartic, arising from the union of two double points (Salmon, Art. 244). The Hessian for this case is

$$\begin{aligned} &- al^2x^6 + (bfh + 2b'b''m - b''^2h - bm^2 - b'^2f)\,y^6 \\ &+ 2(a'l^2 - 2alm)\,x^5y + 2(a'bf + b'm^2 + 2b'b''l - a'b''^2 - b'fh - 2blm)\,xy^5 \\ &+ (afh - a'^2f - 2ab''l + 3hl^2 - 4am^2 + 2a'lm)\,x^4y^2 \\ &+ \begin{pmatrix} abf - ab''^2 + 2a'b'f - 6a'b''m - 3fh^2 \\ + 3hm^2 - 4bl^2 + 2b'lm + 6b''hl \end{pmatrix} x^2y^4 \\ &+ 2(ab'f - a'fh - 2a'm^2 - 2b'l^2 - 2ab''m + 6hlm)\,x^3y^3 \\ &+ 2(bfl + b''m^2 - b''^2l - b''fh)\,y^5 \\ &- 2(2b''lm + 2a'b''f + 3fhm - 3m^3 - 4b'fl)\,y^4x \\ &- 2(2afl - 3l^3)\,x^4y \\ &- 2(ab''f + 2b''l^2 - 3fhl - 6lm^2 + 6a'fm)\,y^3x^2 \\ &- 2(2a'fl + 3afm - 6l^2m)\,x^3y^2 \\ &- (3f^2h - 3fm^2 + 4b''fl)\,y^4 \\ &- 6a'f^2y^3x - 3af^2x^2y^2 - 6f^2ly^3 = 0. \end{aligned}$$

To determine the form of the curve in the neighborhood of the origin we employ the method explained by Salmon (Art. 56).

We write $y = Ax^{\beta}$ and determine β by the condition, that it shall be positive, and that the indices of two or more terms shall be equal, and less than the index of any other term. Having found β, we determine A by equating to zero the quantity multiplying the terms with equal index.

In the present case we find $\beta = 2$; that is

$$y = Ax^2,$$

and the equation for the determination of A is

$$6f^2lA^3 + 3af^2A^2 + 2(2afl - 3l^3)A + al^2 = 0.$$

The roots of this equation are, in general, unequal, since the discriminant

$$12f^2l^2(864afl^6 + 56a^3f^3l^2 - 360a^2f^2l^4 - 432l^8 - 3a^4f^4)$$

does not vanish.

If a_1, a_2, a_3 are the roots of this equation, the curve near the origin resembles the curves $y = a_1x^2$, $y = a_2x^2$, $y = a_3x^2$; i.e. three branches of the Hessian touch each other at the origin, and it should be observed that, since A is determined by a cubic equation, at least one of these branches must be real.

If in the preceding equations a and l have the particular values

$$a = 6fp^2, \quad l = -fp;$$

the origin is a ramphoid cusp or node-cusp on the quartic arising from the union of a double point and a cusp (Salmon, Art. 244).

The equation for the determination of A becomes, after division by $-6f^3p$,

$$A^3 - 3pA^2 + 3p^2A - p^3 = (A - p)^3 = 0.$$

The cubic has three roots equal to p, and to the degree of approximation to which we have yet proceeded, the three branches of the Hessian coincide.

Let
$$y = px^2 + Bx^{\lambda}.$$

Substituting this value of y in the equation of the Hessian, we can make the indices of two or more terms equal, and less than the index of any other term, by taking

$$\lambda = \tfrac{7}{3}.$$

The Hessian, then, near the origin resembles the curve

$$y = px^2 + Bx^{\frac{7}{3}},$$

and therefore consists of three osculating branches, two of which are imaginary.

If in addition to the above values

$$a = 6fp^2, \quad l = -fp;$$

we have $$a' = 3frp, \quad m = -fr;$$
the origin is an oscnode on the quartic arising from the coincidence of three double points as consecutive points of a curve of finite curvature.

And if we further have
$$\begin{aligned} b &= 6f(d^2 - M^2), \\ b' &= 3f(dr - L), \\ b'' &= -3fd; \end{aligned}$$
the origin is a tacnode-cusp on the quartic.

The cubic for the determination of A is the same for the three last cases, and therefore the ramphoid cusp, the oscnode, and the tacnode-cusp give rise to the same singularity on the Hessian.

Let $$c = c' = c'' = f = g = n = 0.$$
The quartic becomes
$$ax^4 + 4a'x^3y + 6hx^2y^2 + 4b'xy^3 + by^4 + 4a''x^3 + 12lx^2y + 12mxy^2 + 4b''y^3 = 0.$$
The origin is a triple point, the tangents at which are given by the equation
$$a''x^3 + 3lx^2y + 3mxy^2 + b''y^3 = 0.$$
The equation of the Hessian is
$$\begin{aligned} u_6 &+ 2(a''l^2 - a''^2m)\,x^5 \\ &+ 2(3l^3 - 2a''lm - a''^2b'')\,x^4y \\ &+ 4(3l^2m - 2a''b''l - a''m^2)\,x^3y^2 \\ &+ 4(lm^2 - 2a''b''m - b''l^2)\,x^2y^3 \\ &+ 2(3m^3 - 2b''lm - a''b''^2)\,xy^4 \\ &+ 2(b''m^2 - b''^2l)\,y^5 = 0. \end{aligned}$$
The origin is then a quintuple point on the Hessian, the tangents at it being given by the equation
$$\begin{aligned} &a''(l^2 - a''m)\,x^5 \\ &+ (3l^3 - a''^2b'' - 2a''lm)\,x^4y \\ &+ 2(3l^2m - a''m^2 - 2a''b''l)\,x^3y^2 \\ &+ 2(3lm^2 - b''l^2 - 2a''b''m)\,x^2y^3 \\ &+ (3m^2 - a''b''^2 - 2b''lm)\,xy^4 \\ &+ b''(m^2 - b''l)\,y^5 = 0. \end{aligned}$$
This equation may be written
$$(a''x^3 + 3lx^2y + 3mxy^2 + b''y^3) \\ \times [(l^2 - a''m)\,x^2 + (lm - a''b'')\,xy + (m^2 - b''l)\,y^2] = 0,$$

which shows that three of the tangents at the quintuple point on the Hessian coincide with the tangents at the triple point on the quartic, the equation of the other two tangents being

$$(l^2 - a''m)\,x^2 + (lm - a''b'')\,xy + (m^2 - b''l)\,y^2 = 0.$$

Let $$a'' = c = c' = c'' = g = 0.$$

The origin is evidently a double point, the tangents being

$$y = 0,$$
$$fy + 2nx = 0.$$

But it will be found that the tangent $y = 0$ meets the curve in four consecutive points, and is therefore a stationary tangent. The origin is, then, a flecnode resulting from the coincidence of a double point and a point of inflexion. The Hessian is

$$\begin{aligned} &u_6 + u_5 \\ &+ (3fm^2 - 3f^2h - 4b''fl - 4bn^2 + 2b''mn + 6b'fn)\,y^4 \\ &+ 2\,(3fhm - 2b'n^2 - 2b''ln - 3a'f^2 + 6m^2n)\,y^3x \\ &+ 3\,(6lmn - 2a'fn - af^2)\,y^2x^2 \\ &+ 2\,(6l^2n - 3afn - 2a'n^2)\,yx^3 - 4an^2x^4 \\ &+ 2\,(6fmn - 3f^2l - 2b''n^2)\,y^3 + 12mn^2y^2x + 12ln^2yx^2 \\ &+ 3fn^2y^2 + 6n^3xy = 0. \end{aligned}$$

The origin is a flecnode on the Hessian, and the tangents at it

$$y = 0,$$
$$fy + 2nx = 0$$

coincide with the tangents at the flecnode on the quartic.

Let $$a'' = b'' = c = c' = c'' = f = g = 0.$$

The axes $x = 0$, $y = 0$ are tangents at a double point, and since each meets the curve in four coincident points they are also stationary tangents. The origin is a bi-flecnode arising from the union of two points of inflexion. The Hessian for this case is

$$u_6 + u_5 + u_4 + 12ln^2x^2y + 12mn^2xy^2 + 6n^3xy = 0.$$

The origin is, then, a bi-flecnode on the Hessian, and the tangents at it coincide with the tangents at the bi-flecnode on the quartic.

ON AN EXTENSION OF HOLDITCH'S THEOREM.

By PROF. W. H. ECHOLS, Rolla, Mo.

The demonstration of Elliott's extension of Holditch's Theorem in polar co-ordinates is interesting on account of the simple form of the differential expression for the *associate* area, and because the sectorial area swept over by the respective *radii vectores* are sometimes more easily investigated than the corresponding areas of rectangular co-ordinates. The theorem can be demonstrated in polar without the aid of the rectangular co-ordinates, but the reduction is more lengthy.

The following is perhaps the simplest method of reaching the result. In the expression (Williamson's Integral Calculus, p. 209)

$$(m+n)^2 y dx = (m+n)(my_1 dx_1 + ny_2 dx_2) - mn(y_2 - y_1)\, d(x_2 - x_1),$$
$$(m+n)^2 x dy = (m+n)(mx_1 dy_1 + nx_2 dy_2) - mn(x_2 - x_1)\, d(y_2 - y_1);$$

write for brevity $\quad m = (m+n)\varkappa' \quad$ and $\quad n = (m+n)\varkappa$,

and substitute

$$\begin{aligned}
y &= \rho \sin\theta, & dy &= \sin\theta\, d\rho + \rho\cos\theta\, d\theta;\\
x &= \rho \cos\theta, & dx &= \cos\theta\, d\rho - \rho\sin\theta\, d\theta;\\
y_1 &= \rho_1 \sin\theta_1, & dy_1 &= \sin\theta_1\, d\rho_1 + \rho_1\cos\theta_1\, d\theta_1;\\
x_1 &= \rho_1 \cos\theta_1, & dx_1 &= \cos\theta_1\, d\rho_1 - \rho_1\sin\theta_1\, d\theta_1;\\
y_2 &= \rho_2 \sin\theta_2, & dy_2 &= \sin\theta_2\, d\rho_2 + \rho_2\cos\theta_2\, d\theta_2;\\
x_2 &= \rho_2 \cos\theta_2, & dx_2 &= \cos\theta_2\, d\rho_2 - \rho_2\sin\theta_2\, d\theta_2;\\
y_2 - y_1 &= \lambda \sin\omega, & d(y_2 - y_1) &= \sin\omega\, d\lambda + \lambda\cos\omega\, d\omega;\\
x_2 - x_1 &= \lambda \cos\omega, & d(x_2 - x_1) &= \cos\omega\, d\lambda - \lambda\sin\omega\, d\omega;
\end{aligned}$$

where λ is the variable length of the moving line, and ω the angle which it makes with the initial line.

Subtracting one equation from the other, we have

$$\rho^2 d\theta = \varkappa\rho_2^2 d\theta_2 + \varkappa'\rho_1^2 d\theta_1 - \varkappa\varkappa'\lambda^2 d\omega;$$

or

$$dP = \varkappa dC' + \varkappa' dC - \varkappa\varkappa' dA,$$

using P, C, C', and A to represent the respective areas.

Now let λ make a finite shift; then

$$\int_{\theta'}^{\theta''} dP = \varkappa \int_{\theta_2'}^{\theta_2''} dC' + \varkappa' \int_{\theta_1'}^{\theta_2''} dC - \varkappa\varkappa' \int_{\omega'}^{\omega''} dA$$

is the general equation connecting the sectorial areas swept over by ρ, ρ_1, ρ_2, and the associate area A, the latter being determined by the curve $\lambda = f(\omega)$.

If we integrate for a complete circuit, we obtain

$$P = xC' + x'C - xx'A,$$

the different cases of which are noticed in Williamson's Calculus.

The mean value of P for $0 < x < 1$, since $x + x' = 1$, is

$$M = C' \int_0^1 x dx + C \int_0^1 x' dx' - A \int_0^1 x(1 - x)\, dx$$

$$= \tfrac{1}{2}(C + C') - \tfrac{1}{6}A,$$

the familiar form of the true prismoid formula for the mean area, where C and C' are the end areas, and A the area of the base of the associate cone; the equation to its boundary being $\lambda = f(\omega)$, where λ is the projection on the base of the curved surface generator, and ω the angle through which it turns. If, however, the extremities and the line oscillate back to their former positions without revolving, then

$$C = 0, \quad C' = 0, \quad \text{and} \quad A = 0,$$

and also $P = 0$, because the positive and negative areas destroy each other in complete circuit, and the point P either retraces its path or crosses it before returning to its starting point, thus dividing the closed area into equal positive and negative portions.

If then, instead of integrating for a complete circuit of P, we integrate only for a complete circuit of any one loop, we obtain the area of that loop (and thus of the whole closed curve, provided none of the loops overlap or have a portion of their area in common), the data being such that we can obtain the limits of θ_1, θ_2, and ω, which correspond to the shift of λ as P describes the loop, the values of $\int dC$, $\int dC'$, and $\int dA$ being obtainable.

SPECIAL FORMS OF THE MOMENTAL ELLIPSOID OF A BODY.

By Prof. S. T. Moreland, Lexington, Va.

In works on analytical mechanics it is shown

1. That for every point within or without a body an ellipsoid may be constructed such that the square of the reciprocal of any semidiameter is equal to the moment of inertia of the body with respect to an axis coincident with this semidiameter. This ellipsoid is known as a momental ellipsoid.

2. That if H' be the moment of inertia of a body with respect to any axis in space, H its moment of inertia with respect to a parallel axis through the centre of mass of the body, p the perpendicular distance between these two axes, M the mass of the body; then

$$H' = H + Mp^2.$$

These two propositions will be used here to discover whether there are any points for which: (a) the momental ellipsoid has its two circular sections at right angles to each other; (b) the momental ellipsoid is a sphere. We are guided in the search by the fact that the momental ellipsoid of every point situated on a normal to a circular section of the central ellipsoid (the momental ellipsoid which has the centre of mass of the body for its centre) has one circular section perpendicular to the normal upon which the point is situated, and we are to find whether for any particular point or points on this normal the other circular section is perpendicular to this one.

Let $g \equiv$ the centre of mass of the body,
$M \equiv$ the mass of the body,
$O \equiv$ one of the points sought on the normal to a circular section of the central ellipsoid,
$d \equiv$ the distance from g to O,
$r \equiv$ any semidiameter of the ellipsoid whose centre is O,
$\rho \equiv$ a semidiameter parallel to r, of the ellipsoid whose centre is g,
$a > b > c \equiv$ the semiaxes of the ellipsoid whose centre is g,
$\alpha \equiv$ the angle between r, or ρ, and the line d,
$\beta \equiv$ the angle between d and the semiaxis c,
$s \equiv$ the semidiameter of the central ellipsoid coincident with d.

The normal to a circular section of the central ellipsoid (at its centre) evidently lies in the plane ac. The angle β is also the angle which the circular sections of the central ellipsoid make with the axis a.

We readily find
$$\cos^2\beta = \frac{a^2(b^2 - c^2)}{b^2(a^2 - c^2)}, \tag{1}$$
or
$$\cos 2\beta = \frac{a^2(b^2 - c^2) - c^2(a^2 - b^2)}{b^2(a^2 - c^2)}; \tag{2}$$
$$\frac{1}{s^2} = \frac{\cos^2\beta}{c^2} + \frac{\sin^2\beta}{a^2}. \tag{3}$$
Propositions 1 and 2 give
$$\frac{1}{r^2} = \frac{1}{\rho^2} + Md^2 \sin^2 \alpha. \tag{4}$$

As already pointed out, the section perpendicular to d, of the ellipsoid whose centre is O, is circular for all values of d. Examine now the section containing the line d and normal to the plane ac.

If $\alpha = 0$, then by hypothesis $\rho = s$, and by (4)
$$\frac{1}{r^2} = \frac{1}{\rho^2} = \frac{1}{s^2};$$
$$\therefore\ r = s.$$
If $\alpha = \frac{1}{2}\pi$, then in like manner $\rho = b$, and by (4)
$$\frac{1}{r^2} = \frac{1}{b^2} + Md^2. \tag{5}$$
If the section is circular, r in this equation must equal s, the value found for r when $\alpha = 0$. That is, d is determined by the condition
$$\frac{1}{s^2} = \frac{1}{b^2} + Md^2,$$
or
$$Md^2 = \frac{1}{s^2} - \frac{1}{b^2}, \tag{6}$$
which (1) and (3) reduce to
$$Md^2 = \frac{1}{a^2} + \frac{1}{c^2} - \frac{2}{b^2}. \tag{7}$$
Remembering that for the section under consideration ρ is the semidiameter of an ellipse whose axes are s and b, and makes an angle α with s, we see that
$$\frac{1}{\rho^2} = \frac{\cos^2\alpha}{s^2} + \frac{\sin^2\alpha}{b^2}. \tag{8}$$
The substitution in (4) of the values of Md^2 and $1/\rho^2$ obtained from equations (6) and (8) gives
$$\frac{1}{r^2} = \frac{1}{s^2};$$

that is, $r = s$ for all values of α. Hence this section is circular, and at right angles to the circular section which is normal to d.

Equation (7) gives two values for d, equal with opposite signs, which shows there are two points, on opposite sides of g, satisfying the imposed condition. There are two points on the normal to the other circular section of the central ellipsoid; hence there are, in all, four points where the two circular sections of the momental ellipsoid are at right angles to each other.

We see from equation (7) that for d to have real values we must have

$$\frac{1}{a^2} + \frac{1}{c^2} > \frac{2}{b^2},$$

which means that β must be less than 45°.

If
$$\frac{1}{a^2} + \frac{1}{c^2} = \frac{2}{b^2}, \quad \text{or} \quad \beta = 45°,$$
$$d = 0,$$

and the four points reduce to one at the centre of the central ellipsoid. The central ellipsoid itself now has its two circular sections at right angles to each other. If a, c, and M are constant, and b variable, which would make β variable, the four points would shift their positions, and their locus is easily found to be the lemniscate of Bernoulli cutting the axis c at right angles. The equation of the lemniscate is

$$d^2 = \frac{a^2 - c^2}{Ma^2c^2} \cos 2\beta.$$

If
$$\frac{1}{a^2} + \frac{1}{c^2} < \frac{2}{b^2},$$

d in equation (7) is imaginary, and there is no point on the normal to the circular sections of the central ellipsoid for which the momental ellipsoid has its two circular sections at right angles to each other.

To find whether the points found above are centres of *ellipsoids* as distinguished from *spheres*, examine a third section, the one made by the plane ac.

ρ makes an angle $\alpha + \beta$ with c, and lies in the plane ac; therefore, as in (3),

$$\frac{1}{\rho^2} = \frac{\cos^2(\alpha + \beta)}{c^2} + \frac{\sin^2(\alpha + \beta)}{a^2}.$$

The substitution in (4) of the values of $1/\rho^2$ and Md^2 obtained from this and equation (7) gives

$$\frac{1}{r^2} = \frac{a^2b^2\cos^2(\alpha + \beta) + b^2c^2\sin^2(\alpha + \beta) + [a^2(b^2 - c^2) - c^2(a^2 - b^2)]\sin^2\alpha}{a^2b^2c^2}. \quad (9)$$

Pass to rectangular co-ordinates with O as a centre, the line d as axis of z, and a line perpendicular to d as axis of x; and we find

$$\begin{aligned} &z^2\left[a^2b^2\cos^2\beta + b^2c^2\sin^2\beta\right] \\ &+ x^2\left[a^2b^2\sin^2\beta + b^2c^2\cos^2\beta + a^2(b^2-c^2) - c^2(a^2-b^2)\right] \\ &- zxb^2(a^2-c^2)\sin 2\beta = a^2b^2c^2. \end{aligned} \tag{10}$$

This is the equation of an ellipse. Its principal axes are found to be inclined $45°$ to z or d. Let c_1 and a_1 be its semiaxes; then

$$c_1 = \frac{abc}{\sqrt{\left[a^2b^2 + b^2c^2 - a^2c^2 + b^2(a^2-c^2)\sin\beta\cos\beta\right]}}, \tag{11}$$

$$a_1 = \frac{abc}{\sqrt{\left[a^2b^2 + b^2c^2 - a^2c^2 - b^2(a^2-c^2)\sin\beta\cos\beta\right]}}. \tag{12}$$

We see that this section of the ellipsoid whose centre is O is an ellipse, and hence the ellipsoid is not a sphere.

To find for what points the ellipsoids are spheres we make $c_1 = a_1$ in equations (11) and (12). This requires

$$\beta = 0;$$
$$\therefore\ b = a;$$
$$\therefore\ c_1 = a_1 = c.$$

Hence the points sought lie on c; there are evidently two such points, one on each side of the centre g. Equation (7) now reduces to

$$\begin{aligned} Md^2 &= \frac{1}{a^2} + \frac{1}{c^2} - \frac{2}{a^2} \\ &= \frac{1}{c^2} - \frac{1}{a^2}; \end{aligned}$$

and in order that d may have a real value, c must be less than a. We conclude, therefore, that if the central ellipsoid be one of revolution about its shortest axis, there are two points on that axis for which the momental ellipsoids are spheres.

The result just reached concerning the sphere may be obtained very simply by analytical methods as in Bartlett's Analytical Mechanics, p. 169. Poinsot in his *Théorie Nouvelle de la Rotation des Corps*, p. 63, gives a simple proof that for a sphere O must be on c, and $c < a = b$; but his proof is not conclusive. In substance, he says that since by hypothesis the ellipsoid centre O is a sphere, every section of it is circular; therefore, from the relation

$$H' = H + Md^2,$$

the section of the central ellipsoid by a plane normal to the line joining its centre

with O, must be circular; therefore, the central ellipsoid must be one of revolution. The last is a *non-sequitur*, because, as we have seen, if O is on the normal to a circular section, it satisfies the condition that the two parallel sections normal to d, one with centre at O and the other with centre at g, the centre of mass of the body, shall both be circular. Poinsot's reasoning shows no more than that O must be on a normal to a circular section of the central ellipsoid; it does not show that this circular section must be a principal section, although he gives that as his conclusion.

It must be confessed that, while it was shown above that under certain conditions there are four points for which the momental ellipsoid has two circular sections at right angles to each other, the analysis does not show conclusively that there are *only* four such points. We have discovered all such points as are situated on normals to central circular sections, but are there no such points elsewhere?

ON THE MASS OF TITAN.

By PROF. ORMOND STONE, University of Virginia, Va.

There is a numerical error in the value of a_6 as given on page 169, Vol. III of this journal. The correct value is approximately $-\frac{1}{2}a_3^2$. The introduction of this in equation (19), p. 166, adds other appreciable terms, the largest of which is $-3a_3a_6 = \frac{3}{2}a_3^3$. If, also, we take into account the term $m'P_0$ in ν, we may, in the place of equation (31), write

$$(2a_3P_0 + P_3)\,m' = \left[1 - 9\left(\frac{n' - n}{n}\right)^2\right]a_3,$$

approximately; whence

$$m' = \frac{1}{4617}.$$

This change in the mass leads, of course, to corresponding changes in the values of the coefficients a_1, n_1, etc.

PRIZES.

R. ACCADEMIA DELLE SCIENZE FISICHE E MATEMATICHE DI NAPOLI.

Sulle curve piane del 4° ordine in relazione con l'interpretazione geometrica delle forme invariantive della forma ternaria biquadratica.

L'Accademia desidera un'esposizione analitica sistematica delle più notevoli proprietà delle curve piane del 4° ordine in relazione con l'interpretazione geometrica delle forme invariantive della forma ternaria biquadratica. La Memoria dovrebbe trattare: 1° delle polari della curva di 4° ordine; 2° delle sue tangenti doppie; 3° dei suoi flessi; 4° dei caratteri analitici invariantivi che distinguono le linee speciali del 4° ordine; 5° della geometria sopra una curva del 4° ordine.

Premio: *lire mille*. Tempo utile: *31 marzo 1889*. Lingue: *italiana, latina, francese*.

Inviare le Memorie (distinte con un motto, il quale dovrà essere ripetuto sopra una busta suggellata che conterrà il nome dell' Autore) al Segretario dell' Accademia.

ACADÉMIE ROYALE DANOISE DES SCIENCES ET DES LETTRES.

I.—Prix: *la Médaille d'or de l'Académie d'une valeur de 320 Couronnes.*

D'après des recherches, en particulier de MM. Weierstrass et Mittag Leffler, on peut développer en séries des fonctions d'une variable avec des zéros et des infinis donnés. Le problème inverse, où il s'agit de trouver les zéros et les infinis de séries données, n'a été résolu que dans des cas très particuliers. Pour provoquer des recherches dans ce sens, l'Académie propose sa médaille d'or comme prix pour la meilleure solution de la question suivante:—

Étant données deux séries quelconques développées suivant des puissances de la variable, avec des coefficients rationnels, et convergente dans toute l'étendue du plan, on demande une méthode qui permette, par un nombre limité de calculs, déterminer une troisième série développée suivant des puissances de la variable, convergente dans toute l'étendue du plan et dont les zéros soient les zéros communs des deux séries données. La méthode devra être éclaircie par des calculs effectués pour un ou plusieurs exemples.

II.—Prix Schou (400 Couronnes).

Une étude des ouvrages de mathématiques arabes qui ont été traduits en latin ou en une langue européenne moderne, principalement de ceux qui traitent de la théorie et de la discussion des équations et de l'application des sections coniques à cette théorie, dans le but de déterminer le degré plus ou moins grand d'originalité dont les

Arabes, dans leurs travaux sur ces matières, font preuve vis-à-vis de leurs devanciers grecs.

Délai fixé: *31 octobre, 1889.* Langues: *latin, français, anglais, allemand, suédois, danois.* Adresser les Mémoires (portant une devise et accompagnés d'un billet caheté muni de la même devise, et renfermant le nom, la profession, et l'adresse de l'Auteur) au Secrétaire de l'Académie, M. *H.-G. Zeuthen* professeur à l'Université de Copenhague.

TEIXEIRA'S INFINITESIMAL ANALYSIS.*

One fourth of this work is occupied by an introduction containing two chapters: one on imaginaries, including their geometrical interpretation; the other on the general theory of functions. The remainder of the work, including about 250 pages, is devoted exclusively to the differential calculus. The method employed is that of infinitesimals, founded upon the method of limits. The fundamental principles are in general clearly stated, and nearly every page gives evidence that the author is well acquainted with the latest results. The work closes with chapters on functions defined by series, singularities of functions, and functions of imaginary variables. [*O. S.*]

*Curso de Analyse Infinitesimal por F. Gomes Teixeira. Porto: Typographia Occidental.

SOLUTIONS OF EXERCISES.

132

A broken line $ABCDE$, . . . etc., is drawn in a plane, having all its angles equal and the concavity always on the same side. Each of the successive parts BC, CD, DE, etc., is half as long as the preceding. The length and direction of AB are given and the common angle. Required the direction and distance from A of the point to which the end of the line approaches, when the construction as described is continued indefinitely. [*Yale Problems.*]

GENERALIZATION.

Let a be the vector AB; m the ratio of the length of any line to the length

of the next; $\pi - \theta$ the common angle; ρ the vector from A to the required point, and λ the angle between ρ and α; m being greater than unity.

We shall have

$$\rho = \alpha + m^{-1}(\cos\theta + \varepsilon\sin\theta)\,\alpha + m^{-2}(\cos\theta + \varepsilon\sin\theta)^2\,\alpha + \ldots.$$

Whence $$(1 - \cos\theta + \varepsilon\sin\theta)\,m\,\rho = \alpha.$$

If it be desired to express ρ in terms of θ, m, and $T\alpha\,(=a)$, we have, by squaring both members and reducing,

$$T\rho = \frac{ma}{\sqrt{(1 + m^2 - 2m\cos\theta)}}.$$

Acting on the same equation with $V(\alpha x)$ we have, after reduction,

$$\tan\lambda = \frac{\sin\theta}{m - \cos\theta}.$$

[*J. N. James.*]

139

RESTATED.

An ellipse touches the given ellipse $a^2y^2 + b^2x^2 - a^2b^2$ at the extremities of a diameter and has its foci on the curve. Find the position of its axis major; and the position of the foci when the angle between the diameter and the axis major of the greater ellipse is a maximum. [*R. H. Graves.*]

SOLUTION.

It is known that, "if any two chords OP, OQ of a conic make equal angles with the tangent at O, the line PQ will cut that tangent in a fixed point." (See Smith's Conic Sections, p. 211).

The fixed point is clearly the pole of the normal at O.

It follows that the required foci are the points of intersection of the given ellipse and a right line joining its centre and the pole of the normal at $O\,(x', y')$.

The equation to this line is

$$\frac{xx'}{a^4} + \frac{yy'}{b^4} = 0. \qquad (1)$$

The chord of contact and (1) are conjugate diameters of the ellipse

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{k^2}. \qquad (2)$$

The angle between them is a maximum when they are equi-conjugate diameters of (2); then,

$$x' = \frac{a^2}{\sqrt{(a^2 + b^2)}} \quad \text{and} \quad y' = \frac{b^2}{\sqrt{(a^2 + b^2)}}.$$

Or, the foci and the points $(\pm x', \pm y')$ are the points of intersection of

$$(a^2 + b^2)(x^2 + y^2) = a^4 + b^4,$$

and

$$b^2x^2 + a^2y^2 - a^2b^2 = 0. \qquad [R. H. Graves.]$$

143

IF a curve of the fourth order have a singular point at which are three coincident tangents, this line passes through the intersection point of the bitangent with the line of junction of the two points of inflexion of the curve. [*F. H. Loud.*]

SOLUTION.

Writing the equation of the curve

$$4l^2y^3 = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4,$$

and putting $u = y/x$, I find, for the points of inflexion,

$$a + 2bu + cu^2 = 0.$$

The co-ordinates x and y are easily expressed in terms of u, and hence the equation

$$x(y' - y'') - y(x' - x'') + x'y'' - x''y' = 0$$

of a line through two given points may be written in terms of the parameters u_1 and u_2 of the two points, and becomes

$$\begin{aligned} &x[a(u_1^4 - u_2^4) + 4bu_1u_2(u_1^3 - u_2^3) + 6cu_1^2u_2^2(u_1^2 - u_2^2) + 4du_1^3u_2^3(u_1 - u_2)] \\ &- y[a(u_1^3 - u_2^3) + 4bu_1u_2(u_1^2 - u_2^2) + 6cu_1^2u_2^2(u_1 - u_2) - cu_1^3u_2^2(u_1 - u_2)] \\ &\qquad\qquad - 4l^2u_1^3u_2^3(u_1 - u_2) = 0, \end{aligned}$$

and when u_1 and u_2 are the roots of the equation, $a + 2bu + cu^2 = 0$, the above becomes

$$(8b^3 - 12abc + 4a^2d)x + (4b^2c - 5ac^2 + a^2e)y - 4a^2l^2 = 0.$$

Also the equation of the curve is written without difficulty in the form

$$\begin{aligned} y^3\{(12a^2bc - 8ab^3 - 4a^3d)x + (9a^2c^2 - 12ab^2c + 4b^4 - a^3e)y + 4a^3l^2\} \\ = [a^2x^2 + 2abxy + (3ac - 2b^2)y^2]^2. \end{aligned}$$

Here the expression in $\{\}$, put equal to zero, is the equation of the bitangent, and comparing this equation with that of the line through the inflexions, it is readily seen that the two lines meet at a point of the axis of x, the singular tangent.

[*F. H. Loud.*]

157

FIND the locus of the instantaneous centre of a tangent to a parabola when one point of the tangent moves in the tangent at the vertex.

[*De Volson Wood.*]

SOLUTION.

The instantaneous centre is the join of the normal and the parallel to the axis from the point where the tangent cuts the tangent at the vertex. This point is the middle point of the normal. If the curve be $y^2 = 2px$, the normal at (x', y') is

$$y - y' = -\frac{y'}{p}(x - x'),$$

and its extremities are (x', y') and $(x' + p, 0)$. Its middle point is

$$x = x' + \tfrac{1}{2}p, \quad y = \tfrac{1}{2}y';$$

accordingly the required locus is

$$4y^2 = 2px - p^2,$$

a parabola whose parameter is $\frac{1}{4}p$ and whose vertex is at the focus of the original curve. [*W. M. Thornton.*]

159 and 160

FIND the centre of gravity of the loop of the Folium of Descartes.

SHOW that the circles of curvature at the node of the Folium of Descartes pass through the middle point of the arc of the loop. [*R. H. Graves.*]

SOLUTION.

The equation of the Folium of Descartes in Cartesian co-ordinates is

$$y^3 - 3axy + x^3 = 0.$$

The form of the equation shows that the curve is symmetrical with respect to an axis which bisects the angle between the original axes. If we transform to this by putting $(x - y)\sqrt{\tfrac{1}{2}}$ and $(x + y)\sqrt{\tfrac{1}{2}}$ in place of x and y, we obtain

$$y^2(b + 3x) = x^2(b - x);$$

$$\therefore y = \pm x \cdot \left(\frac{b - x}{b + 3x}\right)^{\frac{1}{2}},$$

where $$b = \tfrac{3}{2}\sqrt{2} \,.\, a.$$

The ordinate y having values with opposite signs between $x = 0$ and $x = b$, it follows that the centre of gravity of the loop is on the axis of x.

Let dS = an element of the loop's area, and x_1 = the abscissa of the centre of gravity. Hence

$$x_1 = \frac{\int_0^b x\,dS}{\int_0^b dS},$$

and $dS = 2ydx = 2x\left(\frac{b-x}{b+3x}\right)^{\frac{1}{2}}dx$ using the positive value of y. Hence

$$x_1 = \frac{\int_0^b x^2\left(\frac{b-x}{b+3x}\right)^{\frac{1}{2}}dx}{\int_0^b x\left(\frac{b-x}{b+3x}\right)^{\frac{1}{2}}dx}.$$

Let $\left(\frac{b-x}{b+3x}\right)^{\frac{1}{2}} = p$, then x_1 is readily transformed to

$$x_1 = b\frac{\int_1^0 \frac{p^2(1-p^2)^2}{(1+3p^2)^4}dp}{\int_1^0 \frac{p^2(1-p^2)}{(1+3p^2)^3}dp}.$$

But $$\int\frac{p^2(1-p^2)^2}{(1+3p^2)^4}dp = \frac{-p+10p^3-9p^5}{54(1+3p^2)^3} + \frac{1}{54\sqrt{3}}\tan^{-1}(p\sqrt{3}),$$

and $$\int\frac{p^2(1-p^2)}{(1+3p^2)^3}dp = \frac{\frac{1}{3}p^3}{(1+3p^2)^2};$$

$$\therefore\ x_1 = \frac{8\pi b}{27\sqrt{3}} = \frac{8\pi}{9\sqrt{6}}a.$$

To find the radius of curvature of the upper branch of the loop of the node, we have from the equation of the curve

$$y^2 = \frac{x^2(b-x)}{b+3x},$$

and the radius of curvature is

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}};$$

$$\therefore\ \rho = -\frac{(2b^4+8b^3x+12b^2x^3-18x^4)^{\frac{3}{2}}}{4b^3(b+3x)^2}.$$

At the node, when $x = 0$, $\qquad \rho_0 = -b\sqrt{\frac{1}{2}};$

the negative sign denoting that ρ_0 is counted from the node obliquely *downward* for the upper branch of the loop. This being understood, we may disregard the sign, and write for its length merely $\rho_0 = b\sqrt{\frac{1}{2}}$.

We have also
$$\frac{dy}{dx} = \frac{b^2 - 3x^2}{(b - x)^{\frac{1}{2}}(b + 3x)^{\frac{3}{2}}};$$

hence at the node
$$\frac{dy}{dx} = 1;$$

$$\therefore \tan \varphi_0 = 1;$$
$$\therefore \varphi_0 = \tfrac{1}{4}\pi.$$

But the radius ρ_0 is perpendicular to the tangent, and hence makes with the axis of x the angle $-\frac{1}{4}\pi$. If (α, β) are co-ordinates of the centre of curvature, then

$$\alpha = \rho_0 \cos\left(-\tfrac{1}{4}\pi\right) = \tfrac{1}{2}b,$$
$$\beta = \rho \sin\left(-\tfrac{1}{4}\pi\right) = -\tfrac{1}{2}b.$$

Hence the equation of the circle of curvature for the upper branch of the loop is

$$(x - \tfrac{1}{2}b)^2 + (y + \tfrac{1}{2}b)^2 = \tfrac{1}{2}b^2,$$

or
$$x^2 + y^2 + b(y - x) = 0.$$

If this equation be combined with that of the loop, the abscissa of the intersections will be determined by the condition

$$x(x - b) = 0;$$

$\therefore$ $x = 0$, the node for one point,

$x = b$, the abscissa of the middle of the whole arc of the loop.

The radius of curvature for the lower branch of the loop has the same value with the opposite sign, so that the centre of curvature is on the opposite side of the symmetrical axis. It is evident that it will pass through the same points,

$$(x = 0, \quad y = 0), \quad (x = b, \quad y = 0).$$

The area of the loop is $S = \frac{1}{3}b^2$. [*George W. Coakley.*]

163

An elliptic plate of semiaxes a and b, thickness c, and density δ, is revolved at an angular velocity Ω, about a line parallel to its minor axis, and at a fixed distance from it, in a plane through this axis perpendicular to the plane of the plate. Find the straining action on the plate along the line of its minor axis; and the greatest safe value of Ω, if σ be the admissible stress for unit of area on the material of the plate. [*Jas. S. Miller.*]

SOLUTION.

Consider an element of volume of the plate given by the relation

$$dV = y\,dx\,dz.$$

Taking the normal component of the centrifugal force exerted by this elementary

mass, we have for the elementary moment about the axis of the plate

$$dM = \delta\Omega^2 \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}}\, xz dx dz,$$

where z is the distance from the rotation axis to the minor axis of the elliptic lamina. Hence if d be the distance from the fixed axis to the centre line of the plate, we have for the total bending moment

$$M = 2\delta\Omega^2 \frac{b}{a} \int_{d-\frac{1}{2}c}^{d+\frac{1}{2}c} \int_0^a (a^2 - x^2)^{\frac{1}{2}}\, xz dx dz$$

$$= \tfrac{2}{3}\delta\Omega^2 ba^2 dc.$$

In a similar manner we have for the total stress produced by the sum of the components parallel to the plane of the plate,

$$P = 2\delta\Omega^2 \frac{b}{a} \int_{d-\frac{1}{2}c}^{d+\frac{1}{2}c} \int_0^a (a^2 - x^2)^{\frac{1}{2}}\, x dx dz$$

$$= \tfrac{2}{3}\delta\Omega^2 ba^2 c.$$

Hence, since the moment of inertia of the rectangular section is given by

$$I = \tfrac{1}{6}bc^3,$$

we have for the maximum intensity of stress, in absolute units,

$$\sigma = \frac{\delta\Omega^2 a^2}{3c}(c + 6d).$$

If σ be given, then obviously

$$\Omega = \sqrt{\left(\frac{3c\sigma}{\delta a^2 (c + 6d)}\right)}.$$

[*Jas. S. Miller.*]

EXERCISES.

195

UNDER what conditions will the equation

$$(a - \tfrac{1}{2})x^3 + \tfrac{1}{2}x^2 + x - (1 + x)\log(1 + x) = 0$$

have positive real roots? [*R. S. Woodward.*]

196

WHAT values of x and y will render the following value of u a numerical maximum:—

$$\begin{aligned} u = &+ 1.59 \cos(3x + \tfrac{3}{2}y + 67^\circ) \sin \tfrac{3}{2}y \\ &+ 1.16 \cos(6x + 3y + 329^\circ) \sin 3y \\ &+ 0.74 \cos(9x + \tfrac{9}{2}y + 323^\circ) \sin \tfrac{9}{2}y ? \end{aligned}$$

[*R. S. Woodward.*]

197

INTEGRATE
$$\frac{xd^2x}{dt^2} = a\left[b^3 - (b - t)^3\right],$$

a and b being constants. [*R. S. Woodward.*]

198

GIVEN
$$\tan\theta = -\frac{\theta x}{1 - x}.$$

Express θ in a series of ascending powers of x, and show when the series will be converging. [*R. S. Woodward.*]

199

GIVEN
$$ay = \int_{-ab}^{+ab} e^{-x^2} dx.$$

Find y when $a = 0$, without resort to series. [*R. S. Woodward.*]

200

FIND V and $\dfrac{d^2(rV)}{dr^2}$ from the equation

$$\frac{d^n V}{dr^n} = (-1)^{n+1} f(n)\, Mr^{-(n+1)} + \frac{c}{\pi}\,\frac{\sin n\pi}{n},$$

in which c and M are constants, n is zero or any positive integer, and

$$\begin{aligned} f(n) &= 1 \quad \text{for } n = 0, \\ &= n! \quad \text{for } n > 0. \end{aligned}$$

[*R. S. Woodward.*]

201

SHOW that

$$\int_0^\beta \sin x dx \int_0^x \left(\frac{\cos y - \cos\beta}{\cos y - \cos x}\right)^{\frac{1}{2}} dy$$

$$+ \int_\beta^\pi \sin x dx \int_0^\beta \left(\frac{\cos y - \cos\beta}{\cos y - \cos x}\right)^{\frac{1}{2}} dy = 2\pi \sin^2 \tfrac{1}{2}\beta.$$

Show, also, that the first of these two definite integrals is

$$2\,(\sin\beta - \beta\cos\beta). \qquad [\textit{R. S. Woodward.}]$$

202

GIVEN $$\theta\tan\theta = x \quad \text{and } x < 1;$$

find the nth term of the expansion of θ in a series of ascending powers of x, and develop $2x\sqrt{(\theta^2 + x^2)}\,[\theta x + \theta\,(\theta^2 + x^2)]$ to terms of the fourth order in x.

[*R. S. Woodward.*]

203

IN an ellipse whose major and minor axes are AB, DN, foci F, F', centre C, and semi-latus rectum FG; BG intersects CD in O. Prove $CO = AF$.

[*P. H. Ryan.*]

204

FROM the vertices of the triangle ABC are drawn transversals AA', BB', CC', intersecting in O. Prove

$$\frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} > 6. \qquad [\textit{P. H. Ryan.}]$$

205

A TRIANGLE PQR is inscribed in an ellipse, the two sides PQ and PR pass through the foci, and the line QR produced meets the tangent at P in the point S; show that the polar of S, with respect to a concentric circle through P, passes through the centre of curvature of the ellipse at the point P. [*R. H. Graves.*]

206

IF from any point four normals be drawn to each of the hyperbolas,

$$x^2 - y^2 = a^2 \text{ and } xy = k^2,$$

the centres of mean position of the feet of the two sets of normals are coincident.

[*R. H. Graves.*]

207

P and Q are middle points of opposite edges of a tetraedron. A plane through PQ intersects two other opposite edges in M and N. Show that MN is bisected by PQ. [*Asaph Hall.*]

208

PROVE that the surface of an oblate spheroid whose major semiaxis is a and eccentricity e, is equal to

$$2a^2\pi\left[1 + \text{Naperian log}\left(\frac{1+e}{1-e}\right)^{\frac{1-e^2}{2e}}\right].$$

[*R. S. Woodward.*]

209

Show whether the radius vector of a particle describing a curve which is the resultant of any two simple harmonic motions, at right angles to one another and in the same plane, sweeps out areas which are proportional to the times.

[*Jas. S. Miller.*]

210

The length of a bar having the temperature t_0 is l_0. Prove that when the bar rises to the temperature t, the length becomes

$$l = l_0 e^{\varepsilon (t - t_0)},$$

e being the Naperian base of logarithms, and ε the linear coefficient of expansion, assumed constant. If ε is not constant, but a function of t, specify the conditions under which

$$l = l_0 e^{[a (t - t_0) + b (t - t_0)^2 + \ldots]},$$

and

$$l = l_0 [1 + A (t - t_0) + B (t - t_0)^2 + \ldots],$$

hold true, $a, b, \ldots, A, B, \ldots$ being constants. [*R. S. Woodward.*]

211

What would be the height h of the earth's atmosphere, if its density at any height z were given by the formula

$$\rho = \rho_0 h^{-1} (h - z),$$

ρ_0 being the density at the earth's surface? [*R. S. Woodward.*]

212

According to the Gaussian theory of a lens or system of lenses, we have the relation of conjugate distances

$$f_1^{-1} + f_2^{-1} = f^{-1},$$

and

$$s = s_0 f_2 f_1^{-1},$$

in which f is constant, and s and s_0 are any two corresponding linear dimensions of the image and object. Show that

$$\frac{ds}{s} = -\frac{df_1}{f_1} \cdot \frac{f_2}{f}, \quad \text{or} \quad +\frac{df_2}{f_2} \cdot \frac{f_1}{f},$$

and that

$$\frac{\Delta s}{s} = -\left(\frac{f_1 + f_2}{f_1^2}\right) \Delta f_1 + \left(\frac{f_1 + f_2}{f_1^2}\right)^2 (\Delta f_1)^2 + \ldots.$$

[*R. S. Woodward.*]

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